Available online at www.sciencedirect.com

ScienceDirect

Journal of Geometry and Physics 58 (2008) 599–614

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

On quasi-Einstein Cartan type hypersurfaces

Małgorzata Głogowska*

Department of Mathematics, Wrocław University of Environmental and Life Sciences, 50-357 Wrocław, Poland

Received 2 August 2007; received in revised form 5 December 2007; accepted 21 December 2007
Available online 11 January 2008

Dedicated to Professor Dr. Vladislav Viktorovich Goldberg on his 70th birthday

Abstract

We investigate curvature properties of quasi-Einstein Cartan type hypersurfaces in semi-Riemannian space forms.
© 2008 Elsevier B.V. All rights reserved.

MSC: 53B20; 53B25; 53C25

Keywords: Pseudosymmetry type manifold; Cartan type manifold; Quasi-Einstein hypersurface; Akivis–Goldberg type hypersurface; Cartan type hypersurface

1. Introduction

Let (M, g) with $\dim M = n \geq 4$ be a semi-Riemannian manifold such that its curvature tensor R satisfies on $U_C \cap U_S \subset M$

$$R = \phi\, \overline{S} + \mu\, g \wedge S + \eta\, G, \tag{1}$$

where U_C, resp. U_S, is the set of all points of given manifold at which its Weyl conformal curvature tensor C is non-zero, resp. its Ricci tensor S is not proportional to g, and ϕ, μ and η are some functions on $U_C \cap U_S$. According to [11], the condition (1) is called the *Roter type equation*. Consequently, a manifold (M, g), $n \geq 4$, satisfying (1) on $U_C \cap U_S \subset M$ is called a *Roter type manifold* [11]. Obviously, we will consider manifolds (M, g) with non-empty $U_C \cap U_S \subset M$. For precise definitions of the symbols used here we refer to Sections 2 and 3 of this paper. Roter type manifolds were investigated in [23,24,32,35]. We mention that several spacetimes are Roter type manifolds, e.g. some Reissner–Nordström–de Sitter spacetimes [35], as well as some generalized Robertson–Walker spacetimes [24]. There are also other Roter type spacetimes (see [23]). It is known that every Roter type manifold satisfies (see Section 2)

$$S \cdot R = L_1\, \overline{S} + L_2\, g \wedge S + L_3\, G, \tag{2}$$
$$R \cdot R - Q(S, R) = L_4\, Q(g, C), \tag{3}$$
$$S^2 = L_5\, S + L_6\, g, \tag{4}$$

* Fax: +48 713280214.
E-mail address: mglog@ozi.ar.wroc.pl.

0393-0440/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2007.12.012

where $L_1, \dots, L_6$ are some functions on $U_C \cap U_S$. In [29] it was shown that some 4-dimensional semi-Riemannian metrics introduced in [1] satisfy (2)–(4). Based on the results of [6,29], in [11] the notion of Akivis–Goldberg type manifolds was introduced. Namely, a semi-Riemannian manifold (M, g), $n \geq 4$, is said to be an *Akivis–Goldberg type manifold* if (2)–(4) hold on $U_C \cap U_S \subset M$. We refer to [11] for a review of the results on Akivis–Goldberg type manifolds. Obviously, every Roter type manifold is an Akivis–Goldberg type manifold. The converse statement is not true [11]. Thus Akivis–Goldberg type manifolds form a class of manifolds which is an essential extension of the class of Roter type manifolds. A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be a *Cartan type manifold* if on $U_C \cap U_S \subset M$ we have (3) and (4) and

$$S \cdot R = L_0\, R + L_1\, \overline{S} + L_2\, g \wedge S + L_3\, G, \tag{5}$$

where $L_0, \dots, L_6$ are some functions on $U_C \cap U_S$. Obviously, every Akivis–Goldberg type manifold is a Cartan type manifold. The converse statement is not true. Thus Cartan type manifolds form a class of manifolds which is an essential extension of the class of Akivis–Goldberg type manifolds. A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be an *Einstein manifold* if

$$S = \frac{\kappa}{n}\, g \tag{6}$$

on M, where κ is the scalar curvature of (M, g). Einstein manifolds form a subclass of the class of quasi-Einstein manifolds. A semi-Riemannian manifold (M, g), $n \geq 3$, is called a *quasi-Einstein manifold* if at every point $x \in M$ its Ricci tensor S has the form

$$S = \alpha\, g + \epsilon\, w \otimes w, \quad \epsilon = \pm 1, \tag{7}$$

where $w \in T_x^* M$ and $\alpha \in \mathbb{R}$. We also mention that another subclass of quasi-Einstein manifolds form *Ricci-simple manifolds*, i.e. semi-Riemannian manifolds having the Ricci tensor of rank at most one. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of conformally flat spaces. Quasi-Einstein hypersurfaces were studied among others in [12,14,19,21,22]. We refer to [2] for a review of results on quasi-Einstein manifolds. It is easy to check that (7) implies

$$S^2 = (\kappa - (n-2)\alpha)\, S + \alpha\, ((n-1)\alpha - \kappa)\, g. \tag{8}$$

We also note that from Propositions 2.4 and 2.5 of this paper it follows that consideration on Einstein or conformally flat manifolds satisfying (2) (or (5)), (3) and (4) is rather not an interesting question.

A manifold (M, g), $n \geq 3$, is said to be *pseudosymmetric* if at every point of M the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent. This is equivalent to

$$R \cdot R = L_R\, Q(g, R) \tag{9}$$

on $U_R = \{x \in M \mid R - (\kappa/(n-1)n)\, G \neq 0 \text{ at } x\}$, where L_R is some function on U_R. The class of pseudosymmetric manifolds is an extension of the class of *semisymmetric manifolds* ($R \cdot R = 0$). A geometric interpretation of the notion of pseudosymmetry is given in [33]. According to [33], pseudosymmetric manifolds are called *pseudosymmetric in the sense of Deszcz* (see also [34]). A manifold (M, g), $n \geq 3$, is said to be *Ricci-pseudosymmetric* if at every point of M the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent. This is equivalent to

$$R \cdot S = L_S\, Q(g, S) \tag{10}$$

on U_S, where L_S is some function on U_S. The class of Ricci-pseudosymmetric manifolds is an extension of the class of *Ricci-semisymmetric* manifolds ($R \cdot S = 0$) as well as of the class of pseudosymmetric manifolds. A geometric interpretation of the notion of Ricci-pseudosymmetry is given in [34]. According to [34], Ricci-pseudosymmetric manifolds are called *Ricci pseudosymmetric in the sense of Deszcz*. A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be a *manifold with pseudosymmetric Weyl tensor* ([2], Section 5), if at every point of M the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent. This is equivalent to (18) on $U_C \subset M$. Manifolds with pseudosymmetric Weyl tensor we will call *Weyl pseudosymmetric in the sense of Deszcz*.

We say that (9), (10) and (18) are *conditions of pseudosymmetry type* (see e.g. [2]). We mention that spacetimes satisfying some conditions of pseudosymmetry type were classified in [16]. We refer to [2,8] for a review of results on manifolds satisfying conditions of pseudosymmetry type.

Let M be a hypersurface immersed isometrically in a semi-Riemannian manifold (N, g^N). If (1), or (2)–(4), or (3)–(5), hold on $U_C \cap U_S \subset M$ then M is said to be a *Roter type hypersurface* [32], an *Akivis–Goldberg type hypersurface* [11] or a *Cartan type hypersurface* [32], respectively.

In this paper we consider hypersurfaces immersed isometrically in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$ with signature $(s, n+1-s)$, $n \geq 4$, where $c = \widetilde{\kappa}/(n(n+1))$ and $\widetilde{\kappa}$ is the scalar curvature of the ambient space. It is known that on such hypersurface we have [25]

$$R \cdot R - Q(S, R) = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g, C). \tag{11}$$

Thus (3) with $L_4 = -((n-2)\widetilde{\kappa})/(n(n+1))$ holds on $U_C \cap U_S \subset M$. Clearly, if (2) and (4), or (4) and (5), hold on $U_C \cap U_S \subset M$ then M is an Akivis–Goldberg type hypersurface or a Cartan type hypersurface, respectively. As we noted above, every manifold satisfying (7) also fulfills (8), i.e. (4) with $L_5 = \kappa - (n-2)\alpha$ and $L_6 = \alpha\,((n-1)\alpha - \kappa)$. Therefore, if M is a quasi-Einstein hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, and (2), resp. (5), holds on $U_C \cap U_S \subset M$ then M is an Akivis–Goldberg type hypersurface, resp. a Cartan type hypersurface.

Let H be the second fundamental tensor of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$. In [32] (Theorem 3.1) (see also Theorem 3.1 in Section 3.1) it was shown that if the tensor H^2 is expressed by a linear combination of H and g at $x \in U_C \cap U_S \subset M$ then (1) holds at this point. Therefore we restrict our considerations to the set $U_H \subset M$ of all points at which H^2 is not expressed by a linear combination of H and g. It is known that $U_H \subset U_C \cap U_S \subset M$ ([31], p. 366; see also Section 3).

The *Cartan hypersurface* M in the sphere $S^{n+1}(c)$ is a compact minimal hypersurface with constant principal curvatures $-(3c)^{\frac{1}{2}}$, 0, $(3c)^{\frac{1}{2}}$ of the same multiplicity. Therefore we have $U_H = M$. It is known that the Cartan hypersurfaces are tubes of constant radius over the standard Veronese embeddings $i : \mathbb{F}P^2 \to S^{3d+1}(c) \to \mathbb{E}^{3d+2}$, $d = 1, 2, 4, 8$, of the projective plane $\mathbb{F}P^2$ in the sphere $S^{3d+1}(c)$ in a Euclidean space $\mathbb{E}^{3d+2}$, where $\mathbb{F} = \mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{Q}$ (quaternions) or $\mathbb{O}$ (Cayley numbers), respectively [3]. The Cartan hypersurface in $S^4(c)$ is a pseudosymmetric quasi-Einstein manifold satisfying $R \cdot R = \frac{\widetilde{\kappa}}{12}\, Q(g, R)$ ([27], Example 2). Every Cartan hypersurface of dimension $n = 6, 12, 24$ is a non-quasi-Einstein and non-pseudosymmetric Ricci-pseudosymmetric manifold satisfying ([28], Proposition 1)

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, S). \tag{12}$$

From Theorem 4.3 of [13] it follows that every Cartan hypersurface of dimension $n \geq 6$ is a Cartan type hypersurface.

Further, (5) with $L_1 = 0$ is satisfied on $U_H \subset M$ of every Ricci-pseudosymmetric hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, ([13], Theorem 3.2), i.e. on U_H we have (4) and

$$S \cdot R = L_0\, R + L_2\, g \wedge S + L_3\, G. \tag{13}$$

In [32] (Theorem 3.3) it was proved that every Ricci-pseudosymmetric hypersurface is a Cartan type hypersurface. In addition, if M is a Ricci-pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (21) then (13) turns into

$$S \cdot R = L_0\, R + \overline{S} + (L_2 - \alpha)\, g \wedge S + (L_3 + \alpha^2)\, G.$$

Examples of quasi-Einstein Ricci-pseudosymmetric hypersurfaces are given in [19,22].

Our results are related to quasi-Einstein Cartan type hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$. We can say, with respect to the above statements, that we consider quasi-Einstein hypersurface in semi-Riemannian spaces of constant curvature satisfying (5). The main result (see Theorem 6.2) states that at every point $x \in U_H \subset M$ we have: (i)

$$R \cdot R = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R) \tag{14}$$

and

$$C \cdot C = \frac{n-3}{2(n-2)} \left(\frac{\widetilde{\kappa}}{n+1} - \frac{\kappa}{n-1} \right) Q(g, C), \tag{15}$$

κ being the scalar curvature of M, or (ii) (12),

$$R \cdot R - \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R) \neq 0 \tag{16}$$

and the tensors $C \cdot C$ and $Q(g, C)$ are linearly independent at this point, or (iii) (16) and

$$R \cdot S - \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, S) \neq 0, \tag{17}$$

and M is 2-quasi-umbilical on some neighbourhood $U \subset M$ of x and

$$C \cdot C = L_C\, Q(g, C) \tag{18}$$

holds on U, where L_C is some function on this set.

2. Preliminaries

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M, g) be an n-dimensional, $n \geq 3$, semi-Riemannian manifold. We denote by ∇, R, C, S and κ the Levi-Civita connection, the Riemann–Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X, Y) = S(X, Y)$, where $X, Y \in \Xi(M)$ and $\Xi(M)$ is the Lie algebra of vector fields on M. We define the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X, Y)$ and $\mathcal{C}(X, Y)$ of $\Xi(M)$ by $(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y$, $\mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ and

$$\mathcal{C}(X, Y) = \mathcal{R}(X, Y) - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right),$$

where $X, Y, Z \in \Xi(M)$ and A is a symmetric $(0, 2)$-tensor. Now the Riemann–Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the $(0, 4)$-tensor G of (M, g) are defined by $R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4)$, $C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4)$ and $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4)$, where $X, Y, Z, X_1, X_2, \ldots \in \Xi(M)$. For symmetric $(0, 2)$-tensors E and F we define their *Kulkarni–Nomizu product* $E \wedge F$ by

$$\begin{aligned}(E \wedge F)(X_1, X_2, X_3, X_4) = {} & E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) \\ & - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3).\end{aligned}$$

Further, for a symmetric $(0, 2)$-tensor E we define the $(0, 4)$-tensor $\overline{E}$ by $\overline{E} = (1/2)\, E \wedge E$. In particular, we have $\overline{g} = G = (1/2)\, g \wedge g$. Now the Weyl tensor C can be presented in the form

$$C = R - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{(n-2)(n-1)}\, G. \tag{19}$$

We refer to [2,15] for the definition of the tensors: $S \cdot R$, $S \cdot C$, $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$, $C \cdot S$, $Q(g, R)$, $Q(g, C)$, $Q(g, S)$, $Q(S, R)$, and $Q(S, C)$. Furthermore, for symmetric $(0, 2)$-tensors E and F we have (see e.g. [14], Section 3)

$$Q(E, F \wedge E) = -\frac{1}{2}\, Q(F, E \wedge E) = -Q(F, \overline{E}). \tag{20}$$

Proposition 2.1 (*[32], Proposition 2.1*). *Let (M, g), $n \geq 4$, be a semi-Riemannian manifold satisfying (7). Then (4), with $L_5 = \kappa - (n-2)\alpha$ and $L_6 = \alpha\,((n-1)\alpha - \kappa)$, is satisfied on M. Moreover, if (2) and (3) hold on $U_C \cap U_S \subset M$ then (M, g) is an Akivis–Goldberg type manifold.*

Proposition 2.2. *Let (M, g), $n \geq 4$, be a semi-Riemannian manifold. The following conditions are equivalent on $U_S \subset M$: (7) and*

$$\overline{S} = \alpha\, g \wedge S - \alpha^2\, G = \alpha^2\, G + \alpha\epsilon\, g \wedge (w \otimes w). \tag{21}$$

Proof. Our assertion is an immediate consequence of Lemma 3.1 of [30] and (7). □

Proposition 2.3. *If* (M, g), $n \geq 4$, *is a semi-Riemannian manifold satisfying* (7)*. Then on* M *we have*

$$Q(S, g \wedge S) = -\frac{1}{2}\, Q(g, S \wedge S) = -Q(g, \overline{S}) = -\alpha\, Q(g, g \wedge S), \tag{22}$$

$$Q(g, \overline{S}) = -\frac{1}{2}\, Q(S - \alpha\, g, g \wedge S). \tag{23}$$

Proof. (22) is a consequence of (7), (20) and (21). (23) follows immediately from (22). □

Proposition 2.4 (*[32], Proposition 2.3*). *Let* (M, g), $n \geq 4$, *be a semi-Riemannian manifold satisfying* (5)*. Let* V *be a set of all points of* $U_C \cap U_S \subset M$ *at which* (1) *or* (7) *is fulfilled. Then the decomposition of the tensor* $S \cdot R$ *in terms* R, $\overline{S}$, $g \wedge S$ *and* G *is unique on* $U_C \cap U_S \setminus V$.

Let (M, g), $n \geq 4$, be a semi-Riemannian manifold. It is known that the decomposition of the curvature tensor R of (M, g) on $U_C \cap U_S \subset M$ given by (1) is unique ([19], Lemma 3.2). If (1) holds on an open non-empty set $U \subset U_C \cap U_S$ then we say that the Roter type equation is satisfied on this set. If (1) holds on U then on this set we have ([17], Theorem 4.2): (9), with $L_R = (n-2)((\mu/\phi)(\mu - (1/(n-2))) - \eta)$, and

$$R \cdot R - Q(S, R) = \left(L_R + \frac{\mu}{\phi}\right) Q(g, C), \tag{24}$$

$$S^2 = \alpha\, S + \beta\, g, \tag{25}$$

$$S \cdot R = -4(\alpha\phi + \mu)\, \overline{S} - 2(\alpha\mu + \eta + \beta\phi)\, g \wedge S - 4\beta\mu\, G, \tag{26}$$

where $\alpha = \kappa + ((n-2)\mu - 1)\phi^{-1}$ and $\beta = (\mu\kappa + (n-1)\eta)\phi^{-1}$. In addition, we also have (18) and $C \cdot R = L_C\, Q(g, R)$, where $L_C = L_R + (1/(n-2))((\kappa/(n-1)) - \alpha)$. Since (1) implies (24)–(26) we have

Theorem 2.1 (*[11], Theorem 3.1*). *Every Roter type semi-Riemannian manifold* (M, g), $n \geq 4$, *is an Akivis–Goldberg type manifold.*

Proposition 2.5. *On every semi-Riemannian Einstein manifold* (M, g), $n \geq 4$, (4) *and* (5) *are satisfied on* M*. Moreover, if* (M, g) *is pseudosymmetric then* (3) *holds on* M.

Proof. Our proposition is an immediate consequence of (4)–(6) and (19). □

Using the Propositions 2.1 and 2.4 and Theorems 3.3 and 3.4 of [7] we can prove the following

Proposition 2.6. *If* (3) *is satisfied on a conformally flat semi-Riemannian manifold* (M, g), $n \geq 4$, *then* (4) *and* (5) *hold on* M.

We finish this section with the following

Lemma 2.1. *Let* D *be a non-zero symmetric* $(0, 2)$*-tensor at a point* x *of a semi-Riemannian manifold* (M, g), $n \geq 3$. *If the following relation is satisfied at* x

$$D \wedge (w \otimes w) = 0, \tag{27}$$

where $w \in T_x^* M$ *and* $w \neq 0$, *then at this point we have*

$$\text{(a) } D = \rho\, w \otimes w, \quad \rho \in \mathbb{R}, \quad \textit{or} \quad \text{(b) rank } D = 2. \tag{28}$$

Proof. From (27) we have

$$w_h w_k D_{ij} + w_i w_j D_{hk} - w_h w_j D_{ik} - w_i w_k D_{hj} = 0, \tag{29}$$

where w_h and D_{hk} are the local components of w and D, respectively. Further, let X^h be the local components of the vector X at x such that $w_h X^h = 1$ at x.

(i) We assume that rank $D = 1$, i.e. $D = \epsilon\, v \otimes v$, where $v \in T_x^*M$, $\epsilon = \pm 1$. Now (29) takes the form

$$w_h w_k v_i v_j + w_i w_j v_h v_k - w_h w_j v_i v_k - w_i w_k v_h v_j = 0.$$

This by transvection with $X^h X^k$ yields $(v_i + \rho_1\, w_i)(v_j + \rho_1\, w_j) = 0$, where $\rho_1 = v_k X^k$, which implies immediately (28)(a).

(ii) We now assume that rank $D \geq 2$. Transvecting (29) with $X^h X^k$ we get

$$D_{ij} = \beta\, w_i w_j + w_j z_i + w_i z_j, \quad z_i = D_{ih} X^h,\ \beta = -X^h X^k D_{hk}. \tag{30}$$

If $\beta = 0$ then (30) yields

$$D_{ij} = \frac{1}{2}\,((w_i + z_i)(w_j + z_j) - (w_i - z_i)(w_j - z_j)). \tag{31}$$

If $\beta \neq 0$ then (30) yields

$$D_{ij} = \beta \left(w_i + \frac{1}{\beta}\, z_i \right) \left(w_j + \frac{1}{\beta}\, z_j \right) - \frac{1}{\beta}\, z_i z_j. \tag{32}$$

From (31) and (32) it follows that rank $D = 2$ at x, i.e. (28)(b), which completes the proof. $\square$

3. Ricci-pseudosymmetric hypersurfaces

Let M with $\dim M = n \geq 3$ be a connected hypersurface immersed isometrically in a semi-Riemannian manifold (N, g^N). We denote by g the metric tensor induced on M from the metric tensor g^N. Further, we denote by ∇ and ∇^N the Levi-Civita connections corresponding to the metric tensors g and g^N, respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = g^N(\xi, \xi) = \pm 1$. We can present the *Gauss formula* and the *Weingarten formula* of (M, g) in (N, g^N) in the form: $\nabla^N_X Y = \nabla_X Y + \varepsilon\, H(X, Y)\, \xi$ and $\nabla_X \xi = -\mathcal{A}X$, respectively, where X, Y are vector fields tangent to M, H is the second fundamental tensor of (M, g) in (N, g^N), $\mathcal{A}$ is the shape operator and $H^k(X, Y) = g(\mathcal{A}^k X, Y)$, $k \geq 1$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by R and R^N the Riemann–Christoffel curvature tensors of (M, g) and (N, g^N), respectively. The *Gauss equation* of (M, g) in (N, g^N) has the form $R(X_1, \ldots, X_4) = R^N(X_1, \ldots, X_4) + \varepsilon\, \overline{H}(X_1, \ldots, X_4)$, where $\overline{H} = (1/2)\, H \wedge H$ and $X_1, \ldots, X_4$ are vector fields tangent to M. Let the equation $x^r = x^r(y^k)$ be the local parametric expression of (M, g) in (N, g^N), where y^k and x^r are the local coordinates of M and N, respectively, and $a, b, h, i, j, k, l, m \in \{1, 2, \ldots, n\}$ and $p, r, t, u \in \{1, 2, \ldots, n+1\}$. Let $\overline{H}_{hijk} = H_{hk} H_{ij} - H_{hj} H_{ik}$ denote the local components of the tensor $\overline{H}$.

Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, $c = \widetilde{\kappa}/(n(n+1))$, where $\widetilde{\kappa}$ denotes the scalar curvature of the ambient space. Now the Gauss equation reads (see e.g. [12])

$$R_{hijk} = \varepsilon\, \overline{H}_{hijk} + \frac{\widetilde{\kappa}}{n(n+1)}\, G_{hijk}, \tag{33}$$

where G_{hijk} are the local components of the tensor G of M. Contracting (33) with g^{ij} and g^{kh}, respectively, we obtain

$$S_{hk} = \varepsilon\, (tr(H)\, H_{hk} - H^2_{hk}) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g_{hk}, \tag{34}$$

$$\kappa = \varepsilon\, ((tr(H))^2 - tr(H^2)) + \frac{(n-1)\widetilde{\kappa}}{n+1}, \tag{35}$$

respectively, where $tr(H) = g^{hk} H_{hk}$, $tr(H^2) = g^{hk} H^2_{hk}$ and S_{hk} are the local components of the Ricci tensor S of M. Further, we define on M the $(0, 2)$-tensor A by (see Eq. (13) of [15])

$$A = H^3 - tr(H)\, H^2 + \frac{\varepsilon\kappa}{n-1}\, H. \tag{36}$$

Now we can check that on M we have (see Eq. (34) of [11])

$$S \cdot R = 2\, H \wedge A - 4 \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} + \frac{\kappa}{n-1} \right) \left(R - \frac{\widetilde{\kappa}}{n(n+1)}\, G \right) - \frac{2\widetilde{\kappa}}{n(n+1)}\, g \wedge S. \tag{37}$$

We denote by U_H the set of all points of M at which the tensor H^2 is not a linear combination of the metric tensor g and the second fundamental tensor H of M. Using (34) and Theorem 4.1 of [25] we can deduce that $U_H \subset U_C \cap U_S \subset M$. Obviously, on $U_C \cap U_S \setminus U_H$ we have

$$H^2 = \alpha\, H + \beta\, g, \tag{38}$$

where α and β are some functions on $U_C \cap U_S \setminus U_H$. Using (33) and (38) we can verify that

$$R \cdot R = \left(\frac{\widetilde{\kappa}}{n(n+1)} - \varepsilon\, \beta \right) Q(g, R)$$

on $U_C \cap U_S \setminus U_H$ (cf. [21], Proposition 3.1(ii)). Thus (9) holds on this set. Further, we have

Theorem 3.1 (*[32], Theorem 3.1*). *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, and let $U_C \cap U_S \setminus U_H \subset M$ be non-empty. Then the Roter equation holds on this set. Moreover, if $U_H \subset M$ is empty then M is a Roter type hypersurface.*

From this it follows

Corollary 3.1 (*[32], Corollary 3.1*). *Let M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If at every point of $U_C \cap U_S$ the hypersurface M has exactly two distinct principal curvatures then M is a Roter type hypersurface.*

Remark 3.1 (*[32], Remark 3.1*). Let M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If at a point of $U_C \cap U_S \subset M$ there are exactly two distinct principal curvatures then multiplicity of each principal curvature is ≥ 2.

Remark 3.2. (i) (cf. [11], Section 2). Let M be a pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, and let (9) holds on $U_C \cap U_S \subset M$. Then

$$Q\left(S - \left(L_R + \frac{(n-2)\widetilde{\kappa}}{n(n+1)} \right) g,\, R - \frac{\widetilde{\kappa}}{n(n+1)}\, G \right) = 0 \tag{39}$$

on $U_C \cap U_S$ ([10], Eq. 3.8). Let $V \subset U_C \cap U_S$ be the set of all points at which $\operatorname{rank}(S - \alpha\, g) \geq 2$, for any $\alpha \in \mathbb{R}$. Now from (39), in view of Lemma 3.4(ii) of [17], it follows that

$$R = \frac{L}{2} \left(S - \left(L_R + \frac{(n-2)\widetilde{\kappa}}{n(n+1)} \right) g \right) \wedge \left(S - \left(L_R + \frac{(n-2)\widetilde{\kappa}}{n(n+1)} \right) g \right) + \frac{\widetilde{\kappa}}{n(n+1)}\, G$$

on V, where L is some function on V. Thus we see that (1) holds on V. An example of a quasi-Einstein pseudosymmetric hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, was given in [19]. For that hypersurface we have $\operatorname{rank}(S - (\kappa/n)\, g) = 1$ on $U_H \subset M$.
(ii) (cf. [36], Lemma 4.2). If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying at $x \in U_H \subset M$ the condition

$$Q\left(S - \alpha\, g,\, R - \frac{\widetilde{\kappa}}{n(n+1)}\, G \right) = 0$$

then (14) holds at x, where $\alpha \in \mathbb{R}$.

Remark 3.3. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$.
(i) (cf. [5], Proposition 3.2 and Theorem 3.1). On the set $U_H \subset M$ the following conditions are equivalent: (12) and

$$H^3 = tr(H)\, H^2 + \lambda\, H, \tag{40}$$

where λ is some function on U_H. We refer to [22,31] for results on Ricci-pseudosymmetric hypersurfaces.
(ii) We refer to [37,38] for recent results on hypersurfaces satisfying

$$H^3 = tr(H)\, H^2 + \lambda\, H + \mu g,$$

where λ and μ are some functions on U_H.

Proposition 3.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If (12) and (16) hold at $x \in U_H \subset M$ then the tensors $C \cdot C$ and $Q(g, C)$ are linearly independent at this point.*

Proof. In Remark 3.3(i) it was mentioned that (12) and (40) are equivalent at x. Further, from Proposition 4.2 from [36] it follows that

$$(n-2)\, C \cdot C = (n-3)\, Q(S, R) + \beta_5\, Q(g, R) + \beta_6\, Q(S, G) \tag{41}$$

at x, where $\beta_5, \beta_6 \in \mathbb{R}$ are defined by (35) of [36]. We suppose that $C \cdot C = L\, Q(g, C)$, $L \in \mathbb{R}$, holds at x. Now (41), by making use of (19) and (20), gives

$$(n-2)L\, Q(g, R) + L\, Q(S, G) = (n-3)\, Q(S, R) + \beta_5\, Q(g, R) + \beta_6\, Q(S, G),$$

which turns into

$$Q(S, R) + \frac{1}{n-3}(\beta_5 - (n-2)L)\, Q(g, R) + \frac{1}{n-3}(\beta_6 - L)\, Q(S, G) = 0.$$

This, in view of Lemma 4.1 of [36], implies

$$Q\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g,\, R - \frac{\widetilde{\kappa}}{n(n+1)}\, G\right) = 0.$$

But from the last relation, by an application of Remark 3.2(ii), it follows that (14) holds at x, a contradiction. □

4. Quasi-Einstein hypersurfaces

Let M be an quasi-Einstein non-Einstein hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. In the following we will assume that $U_C \cap U_S \subset M$ is non-empty.

Lemma 4.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (7) on $U_C \cap U_S \subset M$.*

(i) *The sets $U_H \subset M$ and $U_C \cap U_S \subset M$ coincide.*

(ii) *At every $x \in U_H$ we have*

$$w^l H_{lk} = \rho\, w_k, \qquad w^k = w_l g^{lk}, \tag{42}$$

where ρ is some function on U_H and H_{kl} and w_l are the local components of H and w at x, respectively.

(iii) *On U_H we have*

$$H^3 = (tr(H) + \rho)\, H^2 + \varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \alpha - \varepsilon\rho tr(H)\right) H + \varepsilon\rho\left(\alpha - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) g, \tag{43}$$

$$\begin{aligned} H^4 \;=\; & \left((tr(H) + \rho)^2 - \rho tr(H) + \varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \alpha\right)\right) H^2 \\ & + \left(\varepsilon(tr(H) + \rho)\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \alpha - \varepsilon\rho tr(H)\right) + \varepsilon\rho\left(\alpha - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)\right) H \\ & + \varepsilon\rho(tr(H) + \rho)\left(\alpha - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) g, \end{aligned} \tag{44}$$

$$\begin{aligned} H^4 \;=\; & (2tr(H)(tr(H) + \rho) + \frac{2(n-1)\varepsilon\widetilde{\kappa}}{n(n+1)} - (tr(H))^2 - \varepsilon(\kappa - (n-2)\alpha))\, H^2 \\ & + \varepsilon tr(H)(\kappa - n\alpha - 2\varepsilon\rho tr(H))\, H + \Bigg(2\varepsilon\rho tr(H)\left(\alpha - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) \\ & - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)^2 + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}(\kappa - (n-2)\alpha) + \alpha((n-1)\alpha - \kappa)\Bigg)\, g, \end{aligned} \tag{45}$$

$$\rho(\rho - tr(H)) = (n-1)\varepsilon\left(\frac{\widetilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} + \alpha\right). \tag{46}$$

Proof. (i) In Section 3 it was stated that for any hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, we have $U_H \subset U_C \cap U_S$. Further, let $x \in U_C \cap U_S \setminus U_H$. Thus at x we have $H^2 = \alpha_1 H + \beta_1 g$, $\alpha_1, \beta_1 \in \mathbb{R}$. Substituting this and (7) into (34) we obtain

$$\varepsilon(tr(H) - \alpha_1)\, H = \left(\alpha + \varepsilon\beta_1 - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) g + \epsilon\, w \otimes w. \tag{47}$$

If $tr(H) - \alpha_1 = 0$ then $\alpha + \varepsilon\beta_1 - ((n-1)\widetilde{\kappa})/(n(n+1)) = 0$ and $w = 0$. Thus (7) reduces to $S = (\kappa/n)\, g$, i.e. $x \in M \setminus U_S$, a contradiction. If $tr(H) - \alpha_1 \neq 0$ then (47) turns into

$$H = \alpha_2\, g + \beta_2\, w \otimes w, \quad \alpha_2, \beta_2 \in \mathbb{R}, \tag{48}$$

i.e. M is quasi-umbilical at x. It is known that (48) implies $C = 0$ at x ([25], Theorem 4.1), i.e. $x \in M \setminus U_C$, again a contradiction. Thus $U_C \cap U_S \setminus U_H$ is empty.

(ii) (cf. the proof of Theorem 2.1 of [22]). Using (7) and (34) we find

$$\alpha\, g_{ij} + \epsilon\, w_i w_j = \varepsilon\,(tr(H)\, H_{ij} - H^2_{ij}) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g_{ij} \tag{49}$$

and after transvection with $H^i_k = g^{il} H_{lk}$ also

$$\alpha\, H_{jk} + \epsilon\, w_i H^i_k w_j = \varepsilon\,(tr(H)\, H^2_{jk} - H^3_{jk}) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, H_{jk}\,. \tag{50}$$

But this implies (42).

(iii) (43) is a consequence of (42), (49) and (50). Further, (43) yields

$$H^4 = (tr(H) + \rho)\, H^3 + \varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \alpha - \varepsilon\rho\, tr(H)\right) H^2 + \varepsilon\rho\left(\alpha - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) H,$$

which by making use of (43) turns into (44). Applying into (8) the identity (2.19) of [11], (34) and (43) we obtain (45). Finally, comparing the right hand sides of (44) and (45) we get (46). The last remark completes the proof. □

Lemma 4.2. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (7) on $U_H \subset M$. Then*

$$\text{(a) } Q(S - \alpha\, g, H \wedge (w \otimes w)) = 0, \qquad \text{(b) } A = \phi\, H + \psi\, w \otimes w, \tag{51}$$

$$S \cdot R = -4\alpha\left(R - \frac{\widetilde{\kappa}}{n(n+1)}\, G\right) - \frac{2\widetilde{\kappa}}{n(n+1)}\, g \wedge S + 2\psi\, H \wedge (w \otimes w), \tag{52}$$

$$\text{(a) } \phi = \varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} + \frac{\kappa}{n-1} - \alpha\right), \qquad \text{(b) } \psi = -\varepsilon\epsilon\rho, \tag{53}$$

on this set, where A and ρ are defined by (36) and (42), respectively.

Proof. Using (20) and (7)we get

$$Q(S - \alpha\, g, H \wedge (w \otimes w)) = \epsilon\, Q(w \otimes w, H \wedge (w \otimes w)) = -\frac{\epsilon}{2}\, Q(H, (w \otimes w) \wedge (w \otimes w)) = 0,$$

i.e. (51)(a). Further

$$H^3 - tr(H)\, H^2 = \varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \alpha\right) H - \varepsilon\epsilon\rho\, w \otimes w \tag{54}$$

on U_H (see the proof of Theorem 2.1 of [22]). Applying (36) into (54) we obtain (51)(b). Now (37), by making use of (51)(b), turns into

$$S \cdot R = 4\phi\, \overline{H} - 4\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} + \frac{\kappa}{n-1}\right)\left(R - \frac{\widetilde{\kappa}}{n(n+1)}\, G\right) - \frac{2\widetilde{\kappa}}{n(n+1)}\, g \wedge S + 2\psi\, H \wedge (w \otimes w).$$

This, by an application of (33), yields (52), which completes the proof. □

Proposition 4.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (7) on $U_H \subset M$. If at $x \in U_H$ the following condition is satisfied*

$$\rho_0\, R = \rho_1\, \overline{S} + \rho_2\, g \wedge S + \rho_3\, G + \rho_4\, H \wedge (w \otimes w), \tag{55}$$

where $\rho_0, \ldots, \rho_4 \in \mathbb{R}$, then ρ_0 vanishes at x or at x we have: $\rho_0 \neq 0$, (14),

$$\text{(a)}\ \frac{\kappa}{n-1} = \frac{\widetilde{\kappa}}{n+1}, \qquad \text{(b)}\ \rho = 0, \qquad \text{(c)}\ S = \frac{\kappa}{n}\, g + \epsilon\, w \otimes w. \tag{56}$$

Proof. We suppose that $\rho_0 \neq 0$ at x. Now (55) turns into

$$R = \bar{\rho}_1\, \overline{S} + \bar{\rho}_2\, g \wedge S + \bar{\rho}_3\, G + \bar{\rho}_4\, H \wedge (w \otimes w),$$

where $\bar{\rho}_0, \ldots, \bar{\rho}_4 \in \mathbb{R}$. The last relation yields

$$\begin{aligned} Q(S - \alpha\, g, R) = {} & \bar{\rho}_1\, Q(S - \alpha\, g, \overline{S}) + \bar{\rho}_2\, Q(S - \alpha\, g, g \wedge S) \\ & + \bar{\rho}_3\, Q(S - \alpha\, g, G) + \bar{\rho}_4\, Q(S - \alpha\, g, H \wedge (w \otimes w)). \end{aligned}$$

Applying in this (20), (22) and (51)(a) we find

$$Q(S - \alpha\, g, R) = \eta\, Q(S, G), \quad \eta = \alpha^2 \bar{\rho}_1 + 2\alpha \bar{\rho}_2 + \bar{\rho}_3,$$

which gives $Q(S - \alpha\, g, R - \eta\, G) = 0$. This, together with (7), implies

$$Q(w \otimes w, R - \eta\, G) = 0. \tag{57}$$

Further, from (57), in view of Lemma 3.4(i) of [17], we obtain at x

$$\sum_{X_1, X_2, X_3} w(X_1)\, (R - \eta\, G)(X_2, X_3, X_4, X_5) = 0,$$

where $X_1, \ldots, X_5 \in T_x M$. Now, using Lemma 2.3 of [20] we get

$$R \cdot R = \frac{\kappa}{(n-1)n}\, Q(g, R), \tag{58}$$

$$R \cdot R - Q(S, R) = -\frac{(n-2)\kappa}{(n-1)n}\, Q(g, C). \tag{59}$$

(11) and (59) lead immediately to (56)(a). Now (58) by (56)(b) turns into (14). Next, from (14) we get easily $R \cdot C = (\widetilde{\kappa}/(n(n+1)))\, Q(g, C)$, which implies

$$\sum_{(X_1, X_2),\, (X_3, X_4),\, (X, Y)} (R \cdot C)(X_1, X_2, X_3, X_4; X, Y) = 0.$$

This, in virtue of Proposition 4.1 of [15], is equivalent to

$$\sum_{(X_1, X_2),\, (X_3, X_4),\, (X, Y)} (R \cdot C - C \cdot R)(X_1, X_2, X_3, X_4; X, Y) = 0.$$

Thus we see that the assumptions of Theorem 2.1 of [22] are fulfilled. In particular, from that theorem we have (see Eq. (16) and (17) of [22])

$$\text{(a)}\ \epsilon\, \|w\|^2 = \frac{\widetilde{\kappa}}{n+1} - \frac{\kappa}{n-1}, \qquad \text{(b)}\ w^k H_{kl} = 0. \tag{60}$$

Evidently, (56)(a) and (60)(a) yield

$$\|w\|^2 = 0. \tag{61}$$

Furthermore, (42) and (60)(b) give (56)(b). From (7), by contraction, we obtain $\kappa - n\alpha = \epsilon\, \|w\|^2$, which by (61) reduces to $\alpha = \kappa/n$. This means that (7) turns into (56)(c). The last remark completes the proof. □

Proposition 4.2. *Let M be a pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If (56)(c) holds on $U_H \subset M$ then on this set we have: (14) and (56)(a),*

$$\overline{S} - \frac{\kappa}{n}\, g \wedge S + \left(\frac{\kappa}{n}\right)^2 G = 0 \tag{62}$$

and an equality of the form (55)

$$(tr(H))^2\, R = \overline{S} - \frac{\kappa}{n}\, g \wedge S + \frac{\kappa}{n}\left(\frac{\kappa}{n} + \frac{(tr(H))^2}{n-1}\right) G + \frac{\epsilon\, tr(H)}{2}\, H \wedge (w \otimes w), \tag{63}$$

where w and ϵ are defined by (56)(c).

Proof. First of all we note that (56)(c) yields $(1/2)\,(S - (\kappa/n)\, g) \wedge (S - (\kappa/n)\, g) = 0$. Now (62) is an immediate consequence of the last relation. It is known that if M is a pseudosymmetric hypersurface then rank $H = 2$ and (14) holds on U_H (e.g. see [5]). Thus in view of Lemma 1.1 of [9] at every point of U_H we have

$$H^3_{ij} = tr(H)\, H^2_{ij} + \lambda\, H_{ij}, \quad \lambda \in \mathbb{R}. \tag{64}$$

Further, (14) and (39) lead to

$$Q\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g,\ R - \frac{\widetilde{\kappa}}{n(n+1)}\, G\right) = 0. \tag{65}$$

If we would have rank $(S - ((n-1)\widetilde{\kappa})/(n(n+1))\, g) \geq 2$ at a point $x \in U_H$ then (65), in view of Lemma 3.4(ii) of [17], implies

$$R - \frac{\widetilde{\kappa}}{n(n+1)}\, G = \overline{\lambda}\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g\right) \wedge \left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g\right), \quad \overline{\lambda} \in \mathbb{R},$$

which turns into (1). But (1) and (56)(c) imply $C = 0$, a contradiction. Thus at every $x \in U_H$ we have

$$S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g = \epsilon_1\, w_1 \otimes w_1, \quad \epsilon_1 = \pm 1,\ w_1 \in T^*_x M.$$

This and (56)(c) lead to $\epsilon = \epsilon_1$, $w = w_1$ and (56)(a) at every $x \in U_H$ (cf. [18], Section 3). Now (34) turns into

$$H^2_{hk} = tr(H)\, H_{hk} - \varepsilon\epsilon\, w_h w_k. \tag{66}$$

Transvecting this with $H^h_{\ j}$ and using (64), in a standard way (for instance, see the proof of Proposition 5.1(iii) of [19]) we get (60)(b). On the other hand, we note that (65) by (33) takes the form

$$Q\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g,\ \overline{H}\right) = 0.$$

This implies that (e.g. see Lemma 3.4(i) of [17])

$$w_l \overline{H}_{hijk} + w_j \overline{H}_{hikl} + w_k \overline{H}_{hilj} = 0 \tag{67}$$

at every point of U_H. Transvecting (67) with $H^h_{\ m}$ and using (60)(b) we get

$$w_l (H^2_{hk} H_{ij} - H^2_{hj} H_{ik}) + w_j (H^2_{hl} H_{ik} - H^2_{il} H_{hk}) = 0.$$

Transvecting this with $H^j_{\ m}$ and using again (60)(b) we find that $H^2_{hk} H^2_{ij} - H^2_{hj} H^2_{ik} = 0$. This, by making use of (66), turns into

$$(tr(H)\, H - \varepsilon\epsilon\, w \otimes w) \wedge (tr(H)\, H - \varepsilon\epsilon\, w \otimes w) = 0,$$

which gives

$$\frac{\varepsilon}{2}(tr(H))^2\, H \wedge H = \frac{\epsilon tr(H)}{2}\, H \wedge (w \otimes w).$$

Finally, applying to this (33) and using (56)(a) and (62) we obtain (63), completing the proof. □

Remark 4.1. Results presented in Examples 4.1 and 5.1 of [14] were applied in [19]. Namely, in Example 5.1 of [19] an example of a quasi-Einstein pseudosymmetric hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (56)(c) on $U_H \subset M$ was given. It is easy to check that $tr(H) \neq 0$ holds at every point of U_H of that hypersurface. Thus by (63) we can state that on U_H we have (55) with non-zero function ρ_0.

5. 2-quasi-umbilical hypersurfaces

A hypersurface M in an $(n+1)$-dimensional Riemannian manifold is said to be *quasi-umbilical*, resp. 2-*quasi-umbilical* [4], at a point $x \in M$ when it has a principal curvature with multiplicity $\geq n-1$, resp. $\geq n-2$, i.e. when the principal curvatures of M at x are given by $\lambda_1, \tau, \ldots \tau$, resp. $\lambda_1, \lambda_2, \tau, \ldots \tau$, where τ occurs $(n-1)$-times, resp. $(n-2)$-times. Further, if M is a hypersurface in an $(n+1)$-dimensional semi-Riemannian manifold then M is called *quasi-umbilical* (e.g. see [25]), resp. 2-*quasi-umbilical* (e.g. see [26]), at a point $x \in M$ when rank $(H - \alpha\, g) = 1$, resp. rank $(H - \alpha\, g) = 2$ hold at x, for some $\alpha \in \mathbb{R}$.

Remark 5.1. It is known that a hypersurface M in a semi-Riemannian conformally flat manifold is quasi-umbilical at a point $x \in M$ if and only if its Weyl tensor vanishes at this point (Theorem 4.1, [25]). Therefore every point of $U_C \subset M$ is a non-quasi-umbilical point of M. It is also known that if M is a 2-quasi-umbilical hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, then (18) holds on $U_C \subset M$ (Theorem 3.1, [26]).

Proposition 5.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If on $U_H \subset M$ we have* rank $(H - \tau\, g) = 2$, *for some function τ, then*

$$H^3 = (tr(H) - (n-3)\,\tau)\, H^2 - \left((n-3)(n-2)\frac{\tau^2}{2} - (n-3)\tau tr(H) + \frac{1}{2}((tr(H))^2 - tr(H^2))\right) H$$
$$+ \tau((n-2)(n-1)\frac{\tau^2}{2} - (n-2)\tau tr(H) + \frac{1}{2}((tr(H))^2 - tr(H^2)))\, g \tag{68}$$

on this set. Moreover, if (7) holds on U_H then

$$\rho = -(n-3)\,\tau, \tag{69}$$
$$\frac{(n-2)(n-1)}{n(n+1)}\,\widetilde{\kappa} - \kappa + 2\alpha = (n-2)(n-1)\varepsilon\,\tau^2, \tag{70}$$
$$\tau\left(\frac{3(n-2)(n-1)}{n(n+1)}\,\widetilde{\kappa} - \kappa - 2(n-3)\alpha\right) = \varepsilon(n-2)\,\tau^2\,((n-1)\tau - 2tr(H)), \tag{71}$$

on this set, where the function ρ is defined by (42).

Proof. We obtain (68) by an application of Lemma 2.1(i) of [11] for the tensor $A = H - \tau\, g$. Now the relations (69)–(71) are an immediate consequence of (35), (43) and (68). □

Proposition 5.2. *Let M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$.*

(i) *Let (7) holds at $x \in U_H \subset M$ and let M has at x exactly three distinct principal curvatures λ_1, λ_2 and λ_3. If $\lambda_1 = \rho$ satisfies (42) then λ_1 is a principal curvature of multiplicity* 1 *and*

$$\lambda_1 = -(p-1)\,\lambda_2 - (n-p-2)\,\lambda_3 \tag{72}$$

at this point, where p and $n-p-1$ is multiplicity of λ_2 and λ_3, respectively.

(ii) *Let M has at $x \in U_H \subset M$ three distinct principal curvatures λ_1, λ_2, λ_3 with multiplicity* 1, *p and $n-p-1$, respectively. If (72) holds at x then*

$$\text{rank}\left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} + \lambda_2\lambda_3\right) g\right) = 1, \tag{73}$$

i.e. M is quasi-Einstein at this point.

Proof. (i) Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis at $T_x M$ such that $H(e_i, e_i) = \lambda_i$ and $H(e_i, e_j) = H_{ij} = 0$ for $i \neq j$, where $i, j \in \{1, 2, \ldots, n\}$. Now (49) implies

$$\text{(a) } \psi + \epsilon\, w_i^2 = (tr(H) - \lambda_i)\lambda_i, \qquad \text{(b) } w_i w_j = 0, \quad \text{for } i \neq j, \tag{74}$$

where $\psi = \alpha - ((n-1)\widetilde{\kappa})/(n(n+1))$. Without loss of generality we can assume that $w_1 \neq 0$. Thus (74)(b) implies $w_2 = w_3 = \cdots = w_n = 0$. Using this and (74)(a) we find that

$$\begin{aligned} &\text{(a) } \psi + \epsilon\, w_1^2 = (tr(H) - \lambda_1)\lambda_1, \qquad \text{(b) } n\psi + \epsilon\, w_1^2 = (tr(H))^2 - tr(H^2),\\ &\text{(c) } \psi = (tr(H) - \lambda_2)\lambda_2, \qquad \text{(d) } \psi = (tr(H) - \lambda_3)\lambda_3. \end{aligned} \tag{75}$$

From (74)(a) and (75)(a) it follows that λ_1 has multiplicity 1. Furthermore, from (75)(c) and (75)(d) it follows that $(\lambda_2 - \lambda_3)(tr(H) - \lambda_2 - \lambda_3) = 0$. Thus we have (72), $tr(H) = \lambda_2 + \lambda_3$ and $\psi = \lambda_2\lambda_3$, which completes the proof of (i).

(ii) First of all we note that (34) holds at x, where $\varepsilon = 1$. Further, let $e_1, e_2, \ldots, e_n$ be an orthonormal basis at $T_x M$ such that

$$\begin{aligned} &H(e_i, e_j) = H_{ij} = 0, \quad \text{for } i \neq j,\\ &H(e_1, e_1) = H_{11} = \lambda_1,\\ &H(e_2, e_2) = H_{22} = \lambda_2, \ldots, H(e_{p+1}, e_{p+1}) = H_{p+1\, p+1} = \lambda_2,\\ &H(e_{p+2}, e_{p+2}) = H_{p+2\, p+2} = \lambda_3, \ldots, H(e_n, e_n) = H_{nn} = \lambda_3, \end{aligned}$$

where $i, j \in \{1, 2, \ldots, n\}$. Using this and (72) we get

$$\begin{aligned} &H_{ij}(tr(H) - H_{ij}) = 0, \quad \text{for } i \neq j,\\ &H_{11}(tr(H) - H_{11}) = \lambda_1(p\lambda_2 + (n-p-1)\,\lambda_3),\\ &H_{22}(tr(H) - H_{22}) = \cdots = H_{nn}(tr(H) - H_{nn}) = \lambda_2\lambda_3. \end{aligned}$$

Applying this into (34) we get (73), which completes the proof of (ii). □

6. Quasi-Einstein Cartan type hypersurfaces

Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (5) and (7) on $U_H \subset M$. We define on U_H the (0, 2)-tensor D by

$$D = \epsilon\left(\alpha L_1 + L_2 + \frac{2\widetilde{\kappa}}{n(n+1)}\right) g - 2\psi\, H, \tag{76}$$

where ψ is defined by (53)(b), i.e. $\psi = -\varepsilon\epsilon\rho$. We note that rank $D \neq 1$ at every point of U_H. Indeed, if rank $D = 1$ at a point $x \in U_H$ then M is quasi-umbilical at x which by Remark 5.1 is equivalent to the fact that the Weyl tensor C of M vanishes at x, a contradiction. We now consider the quasi-Einstein hypersurfaces satisfying (2), i.e. the special case of (5).

Proposition 6.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (2) and (7) on $U_H \subset M$. Then at every $x \in U_H$ we have: $\alpha \neq 0$ and (14), or $\alpha = 0$, $L_3 = 0$,*

$$\text{(a) } S = \epsilon\, w \otimes w, \qquad \text{(b) } D \wedge (w \otimes w) = 0. \tag{77}$$

Proof. We note that (2) and (52) yield

$$-4\alpha\, R = L_1\, \overline{S} + \left(L_2 + \frac{2\widetilde{\kappa}}{n(n+1)}\right) g \wedge S + \left(L_3 - \frac{4\alpha\widetilde{\kappa}}{n(n+1)}\right) G - 2\psi\, H \wedge (w \otimes w). \tag{78}$$

If $\alpha \neq 0$ at x then, in view of Proposition 4.1, (14) holds at x. Now we consider the case $\alpha = 0$ at x. Thus (7) reduces to (77)(a). Further, (78) turns into

$$\left(L_2 + \frac{2\widetilde{\kappa}}{n(n+1)}\right) g \wedge S + L_3\, G = 2\psi\, H \wedge (w \otimes w).$$

This, by (77)(a), yields

$$\left(\epsilon\left(L_2+\frac{2\widetilde{\kappa}}{n(n+1)}\right) g-2\psi\, H\right)\wedge(w\otimes w)=-L_3\, G, \tag{79}$$

which gives

$$Q\left(w\otimes w, \left(\epsilon\left(L_2+\frac{2\widetilde{\kappa}}{n(n+1)}\right) g-2\psi\, H\right)\wedge(w\otimes w)\right)=-L_3\, Q(w\otimes w, G).$$

From this, by making use of (20), we obtain

$$-\frac{1}{2}\, Q\left(\left(\epsilon\left(L_2+\frac{2\widetilde{\kappa}}{n(n+1)}\right) g-2\psi\, H\right), (w\otimes w)\wedge(w\otimes w)\right)=-L_3\, Q(w\otimes w, G),$$

which reduces to $L_3\, Q(w\otimes w, G)=0$. From the last equation it follows that $L_3=0$ at x. Applying this, (76) and (77)(a) into (79) we get (77)(b). Our proposition is thus proved. □

Proposition 6.2. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n\geq 4$, satisfying (5) and (7) on $U_H\subset M$. Then at every $x\in U_H$ we have:*
(i) (14) *and*

$$L_0+4\alpha\ \neq\ 0, \tag{80}$$

or (ii) (77)(b) *and*

$$\text{(a) } L_0+4\alpha=0,\qquad \text{(b) } \alpha^2\, L_1+2\alpha\, L_2+L_3=0. \tag{81}$$

Proof. We note that (5) and (52) yield

$$(L_0+4\alpha)\, R+L_1\,\overline{S}+\left(L_2+\frac{2\widetilde{\kappa}}{n(n+1)}\right) g\wedge S+\left(L_3-\frac{4\alpha\widetilde{\kappa}}{n(n+1)}\right) G-2\psi\, H\wedge(w\otimes w)=0. \tag{82}$$

If (80) holds at x then, in view of Proposition 4.1, at this point (14) is satisfied. We assume now that (81)(a) holds at x. Now (82) takes the form

$$L_1\,\overline{S}+\left(L_2+\frac{2\widetilde{\kappa}}{n(n+1)}\right) g\wedge S+\left(L_3-\frac{4\alpha\widetilde{\kappa}}{n(n+1)}\right) G-2\psi\, H\wedge(w\otimes w)=0. \tag{83}$$

Further, (51)(a) and (83) give

$$L_1\, Q(S-\alpha\, g, \overline{S})+\left(L_2+\frac{2\widetilde{\kappa}}{n(n+1)}\right) Q(S-\alpha\, g, g\wedge S)+\left(L_3-\frac{4\alpha\widetilde{\kappa}}{n(n+1)}\right) Q(S-\alpha\, g, G)=0,$$

which by (23) turns into

$$Q(S-\alpha\, g, \Phi_1 g\wedge S+\Phi_2\, G)=0, \tag{84}$$

$$\Phi_1=\frac{\alpha L_1}{2}+L_2+\frac{2\widetilde{\kappa}}{n(n+1)},\qquad \Phi_2=L_3-\frac{4\alpha\widetilde{\kappa}}{n(n+1)}. \tag{85}$$

Now (84), by making use of (7) and (51)(a), reduces to $(2\alpha\,\Phi_1+\Phi_2)\, Q(w\otimes w, G)=0$, which implies

$$\Phi_2=-2\alpha\,\Phi_1. \tag{86}$$

This by (85) leads to (81)(b). Applying (21), (85) and (86) into (83) we obtain (77)(b), which completes the proof. □

Theorem 6.1. *Let M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n\geq 4$, satisfying (5) and (7) on $U_H\subset M$. Then (12) and (16) hold at every point of U_H.*

Proof. We suppose that (80) holds at a point $x \in U_H$. Thus, in view of Proposition 6.2, (14) is satisfied at x. Further, from Theorem 5.1 of [10] it follows that rank $H = 2$ at x, i.e. M has the principal curvatures $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 = \cdots = \lambda_n = 0$. Now, in view of Proposition 5.2(i), we conclude that $\lambda_1 = 0$, a contradiction. Therefore (16) and (77)(b) and (81)(a) hold at x. We can assume that the local components of the covector w, defined by (7), are the following $w_1 \neq 0$, $w_2 = \cdots = w_n = 0$. Now (77)(b) takes the form $D_{ij} = 0$, which by (76) and Remark 5.1 implies $\rho = 0$. Thus (43) reduces to (40). But this, in view of Remark 3.3(i), is equivalent to (12). Our theorem is thus proved. □

Proposition 6.3. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying at $x \in U_H \subset M$ the conditions:* (5), (7) *and* (16) *and let D be the tensor defined by* (76). *Then at x we have: $D = 0$ or* rank $D = 2$.

Proof. From Proposition 6.2 it follows that (77)(b) holds on U_H. Since rank $D \neq 1$, our assertion is an immediate consequence of Lemma 2.1. □

Theorem 6.2. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying* (5) *and* (7) *on $U_H \subset M$. Then at every $x \in U_H$ we have:*

(i) (14) and (15) *hold at x, or*
(ii) (12) and (16) *hold at x and the tensors $C \cdot C$ and $Q(g, C)$ are linearly independent at this point, or*
(iii) (17) *holds at x and M is 2-quasi-umbilical on some neighbourhood $U \subset M$ of x and* (18) *is satisfied on this set.*

Proof. (i) Let rank $H = 2$ at x. It is known that this is equivalent to (14) (e.g. see Theorem 3.2 of [36]). In addition, in view of Proposition 4.3 of [36], (15) holds at x.
(ii) Let rank $H > 2$ and (12) holds at x. Clearly, rank $H > 2$ implies (16) at x. Now, in view of Proposition 3.1, $C \cdot C$ and $Q(g, C)$ are linearly independent at x.
(iii) Let rank $H > 2$ and (17) hold at x. If the tensor D, defined by (76), vanishes at x then $\psi = -\varepsilon\epsilon\rho = 0$, and in a consequence $\rho = 0$ at x. Thus (43) reduces to (40), which by Remark 3.3(i) is equivalent to (12), a contradiction. Therefore D is non-zero at x. Now, in view of Proposition 6.3, on some neighbourhood U of x we have rank $D = 2$. This, together with (76), implies rank $(H - \tau g) = 2$ on U, where τ is some function on U. Now, in view of Theorem 3.1 of [26], (18) holds on U. The last remark completes the proof. □

Acknowledgments

The author is supported by a grant from the Ministerie van de Vlaamse Gemeenschap (Belgium) and the grant 293/GW/2006 of the Wrocław University of Environmental and Life Sciences (Poland).

References

[1] M.A. Akivis, V.V. Goldberg, Semiintegrable almost Grassmann structures, Differential Geom. Appl. 10 (1999) 257–293.
[2] M. Belkhelfa, R. Deszcz, M. Głogowska, M. Hotloś, D. Kowalczyk, L. Verstraelen, A review on pseudosymmetry type manifolds, in: Banach Center Publ., vol. 57, Inst. Math. Polish Acad. Sci., 2002, pp. 179–194.
[3] T.E. Cecil, P.J. Ryan, Tight and Taut Immersions of Manifolds, Pitman Publ. Inc., Boston, London, Melbourne, 1988.
[4] B.-Y. Chen, L. Verstraelen, Codimension 2 submanifolds with a quasi-umbilical normal direction, J. Korean Math. Soc. 13 (1976) 87–97.
[5] F. Defever, R. Deszcz, P. Dhooghe, L. Verstraelen, Ş. Yaprak, On Ricci-pseudosymmetric hypersurfaces in spaces of constant curvature, Results Math. 27 (1995) 227–236.
[6] F. Defever, R. Deszcz, M. Głogowska, V.V. Goldberg, L. Verstraelen, A class of four-dimensional warped products, Demonstr. Math. 35 (2002) 853–864.
[7] R. Deszcz, On conformally flat Riemannian manifolds satisfying certain curvature conditions, Tensor (N.S.) 49 (1990) 134–145.
[8] R. Deszcz, On pseudosymmetric spaces, Bull. Soc. Belg. Math., Ser. A 44 (1992) 1–34.
[9] R. Deszcz, On certain classes of hypersurfaces in spaces of constant curvature, in: Geometry and Topology of Submanifolds, vol. VIII, World Sci., River Edge, NJ, 1996, pp. 101–110.
[10] R. Deszcz, On pseudosymmetric hypersurfaces in spaces of constant curvature, Tensor (N.S.) 58 (1997) 253–269.
[11] R. Deszcz, On some Akivis–Goldberg type metrics, Publ. Inst. Math. (Beograd) (N.S.) 74 (84) (2003) 71–83.
[12] R. Deszcz, M. Głogowska, Examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces, Colloq. Math. 94 (2002) 87–101.
[13] R. Deszcz, M. Głogowska, Some nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces, Publ. Inst. Math. (Beograd) 72 (86) (2002) 81–93.
[14] R. Deszcz, M. Głogowska, M. Hotloś, Z. Şentürk, On certain quasi-Einstein semisymmetric hypersurfaces, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 41 (1998) 151–164.

[15] R. Deszcz, M. Głogowska, M. Hotloś, L. Verstraelen, On some generalized Einstein metric conditions on hypersurfaces in semi-Riemannian space forms, Colloq. Math. 96 (2003) 149–166.
[16] R. Deszcz, S. Haesen, L. Verstraelen, Classification of space-times satisfying some pseudo-symmetry type conditions, Soochow J. Math. 30 (2004) 339–349 (special issue in honor of Professor Bang-Yen Chen).
[17] R. Deszcz, M. Hotloś, On a certain subclass of pseudosymmetric manifolds, Publ. Math. Debrecen 53 (1998) 29–48.
[18] R. Deszcz, M. Hotloś, On some pseudosymmetry type curvature condition, Tsukuba J. Math. 27 (2003) 13–30.
[19] R. Deszcz, M. Hotloś, On hypersurfaces with type number two in space forms, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 46 (2003) 19–34.
[20] R. Deszcz, M. Hotloś, Z. Şentürk, On the equivalence of the Ricci-pseudosymmetry and pseudosymmetry, Colloq. Math. 79 (1999) 211–227.
[21] R. Deszcz, M. Hotloś, Z. Şentürk, Quasi-Einstein hypersurfaces in semi-Riemannian space forms, Colloq. Math. 89 (2001) 81–97.
[22] R. Deszcz, M. Hotloś, Z. Şentürk, On curvature properties of certain quasi-Einstein hypersurfaces, Dept. Math., Agricultural Univ. Wrocław, Ser. A, Theory and Methods, Report No. 104, 2002.
[23] R. Deszcz, D. Kowalczyk, On some class of pseudosymmetric warped products, Colloq. Math. 97 (2003) 7–22.
[24] R. Deszcz, M. Scherfner, On a particular class of warped products with fibres locally isometric to generalized Cartan hypersurfaces, Colloq. Math. 109 (2007) 13–29.
[25] R. Deszcz, L. Verstraelen, Hypersurfaces of semi-Riemannian conformally flat manifolds, in: Geometry and Topology of Submanifolds, vol. III, World Sci., River Edge, NJ, 1991, pp. 131–147.
[26] R. Deszcz, L. Verstraelen, Ş. Yaprak, On 2-quasi-umbilical hypersurfaces in conformally flat spaces, Acta Math. Hungarica 78 (1998) 45–57.
[27] R. Deszcz, L. Verstraelen, Ş. Yaprak, Pseudosymmetric hypersurfaces in 4-dimensional spaces of constant curvature, Bull. Inst. Math. Acad. Sinica 22 (1994) 167–179.
[28] R. Deszcz, Ş. Yaprak, Curvature properties of Cartan hypersurfaces, Colloq. Math. 67 (1994) 91–98.
[29] M. Głogowska, Curvature properties of some four-dimensional manifolds, Demonstr. Math. 34 (2001) 901–918.
[30] M. Głogowska, Semi-Riemannian manifolds whose Weyl tensor is a Kulkarni–Nomizu square, Publ. Inst. Math. (Beograd) (N.S.) 72 (86) (2002) 95–106.
[31] M. Głogowska, On a curvature characterization of Ricci-pseudosymmetric hypersurfaces, Acta Math. Scientia 24B (2004) 361–375.
[32] M. Głogowska, Curvature conditions on hypersurfaces with two distinct principal curvatures, in: Banach Center Publ., vol. 69, Inst. Math. Polish Acad. Sci., 2005, pp. 133–143.
[33] S. Haesen, L. Verstraelen, Properties of a scalar curvature invariant depending on two planes, Manuscripta Math. 122 (2007) 59–72.
[34] B. Jahanara, S. Haesen, Z. Şentürk, L. Verstraelen, On the parallel transport of the Ricci curvatures, J. Geom. Phys. 57 (2007) 1771–1777.
[35] D. Kowalczyk, On the Reissner–Nordström-de Sitter type spacetimes, Tsukuba J. Math. 30 (2006) 263–281.
[36] K. Sawicz, Hypersurfaces in spaces of constant curvature satisfying some Ricci-type equations, Colloq. Math. 101 (2004) 183–201.
[37] K. Sawicz, On some class of hypersurfaces with three distinct principal curvatures, in: Banach Center Publ., vol. 69, Inst. Math. Polish Acad. Sci., 2005, pp. 145–156.
[38] K. Sawicz, On curvature characterization of some hypersurfaces in spaces of constant curvature, Publ. Inst. Math. (Beograd) (N.S.) 79 (93) (2006) 95–107.